ON OPTIMUM CONTROL IN THE PRESENCE OF RANDOM DISTURBANCES

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Considered is a problem of optimum control under the condition of minimum expected decay time of a transient process. A method is described for applying the Liapunov function to this problem [1,2]. Assumed statement of the problem is generalized to include some problems of optimum application of high speed action on systems subject to random disturbances. Discussed are approximate methods for synthesizing optimum control.

Author notes that he discussed the theme of this work and the methods of solution of the considered problems with N.G. Chetaev who gave him a number of valuable suggestions. Especially, detailed remarks were made by Chetaev regarding the application of the Liapunov function to the problems of the investigated systems subject to random disturbances.

1. Statement of problem. Let us consider the system described by the equations

$$\frac{dx}{dt} = Ax + Bu + c\eta \tag{1.1}$$

where x is an n-dimensional vector of phase coordinates of the system, A, B are $n \times n$-matrices, $\eta(t)$ is a random scalar function, u is the n-dimensional control vector. For the given initial conditions x_0, η_0, t_0 it is required to derive the rule for choosing control u^0 which ensures minimum time of decay for the transient process $x(x_0, \eta_0, t_0, t, \eta, u)$ in system (1.1). Depending upon the character of information about the process action, several versions of formulation are possible.

Let us denote by $g(\vartheta)$ the realization of the random function $\eta(t)$ for $t \leqslant \vartheta < t$. A great number (almost all) realizations $g(\vartheta)$ we shall denote by $\Omega(t)$.

Definition 1.1. Let us consider the operators $u[t, g]$ which compare (for fixed $t \geqslant t_0$) with $g(\vartheta)$ the vectors u. The totality U_t of operators

$u[t, g]$ ($t_0 \leqslant t < \infty$, $g \in \Omega(t)$ we shall call t-control.

Vector-function $u(x, \eta, t) = U_x$ which compares (for fixed t) with vectors $x \in G$ and numbers $\eta \in H$ the vectors u, we shall call x-control in the region $G \times H$.

Let us denote

$$\|y\| = (y_1^2 + \ldots + y_n^2)^{1/2}, \qquad \|y\|_k = (y_1^2 + \ldots + y_k^2)^{1/2}$$

Definition 1.2. Let there be given a number $\epsilon > 0$, a natural number $k < n$ and the initial conditions x_0, η_0, t_0. Let us agree to call U_t (t-control) ϵ-permissible on $\{x_1, \ldots, x_k\}$ for the initial conditions x_0, η_0, t_0, if these conditions are satisfied:

1) norm $\|u\|$ satisfies inequality

$$\|u[t, g]\| \leqslant 1, \qquad t \in [t_0, \infty), \qquad g \in \Omega \tag{1.2}$$

2) inequality

$$T[U_t, k, \varepsilon, x_0, \eta_0, t_0] = \int_{t_0}^{\infty} p[U_t, k, \varepsilon, x_0, \eta_0, t_0, t]\, dt < \infty \tag{1.3}$$

is satisfied, where the symbol $p[U_t, k, \epsilon, x_0, \eta_0, t_0, t]$ denotes the probability of inequality

$$\|x(x_0, \eta_0, t_0, t, \eta, u)\|_k > \varepsilon \tag{1.4}$$

for $t > t_0$ along the random solution $x(t)$ of system (1.1), caused by the random functions $\eta(t)$ and $u(t) = u[t, \eta(\vartheta)]$. The set of ϵ-permissible t-controls (for given conditions k, ϵ, x_0, η_0, t_0) we shall denote by $M_t[k, \epsilon, x_0, \eta_0, t_0]$.

The meaning of Definition 1.2 is as follows: ϵ-permissible t-control (totality U_t of operators $u[t, g]$) determines the rule for choosing the $u(t)$ control on the basis of information about realization $g(\vartheta)$ of a random function $\eta(t)$ for $t_0 \leqslant \vartheta \leqslant t$, whereupon this control ensures the decay of the transient process according to $\|x(t)\|_k$ up to $\epsilon \geqslant 0$ with probability arbitrarily close to unity for sufficiently large t.

Definition 1.3. Let there be given a number $\epsilon \geqslant 0$, a natural number $k < n$, regions G_0 and G of the phase-space $\{x\}$ and a set of numbers H. Function U_x, the x-control in the region $G \times H$, we shall call ϵ-permissible on $\{x_1, \ldots, x_k\}$ for initial disturbances x_0 from region G_0, if the following conditions are fulfilled:

1) function $\eta(t)$ may take values only from H for all realizations $g(\vartheta) \in \Omega(t)$, $t_0 \leqslant t < \infty$;

2) solutions of system 1.1 (for $u = u[x, \eta, t]$) with the initial conditions $x_0 \in G_0$ for all realizations $\eta = g(\vartheta)$ remain in region G for $t \geqslant t_0$;

3) norm $\| u \|$ satisfies condition

$$\|u[x, \eta, t]\| \leqslant 1, \quad x \in G, \quad \eta \in H, \quad t \geqslant t_0 \tag{1.5}$$

4) inequality (1.3) is fulfilled (where U_t is replaced by U_x) for all $x_0 \in G_0$, $\eta \in H$, whereupon the trajectory $x(x_0, \eta_0, t_0, t, \eta, u)$ is caused by the control $u(t) = u(x(t), \eta(t), t)$.

The set of ϵ-permissible x-controls we shall denote by $M_x[k, \epsilon, G_0, G]$.

(*Notes 1.1.* In those cases where there can be no misunderstanding we shall use abbreviated terms and notations, omitting complementary characteristics (e.g. permissible control instead of ϵ-permissible t-control, $T[U]$ instead of $T[U_t, k, \epsilon, x_0, \eta_0, t_0]$ etc.).

1.2. Admissibility conditions for U_t(or U_x) must include a requirement for the existence of solutions of system (1.1) for $u(t) = u[t, g]$ or $u = u[x, \eta, t]$ for almost all realizations $\eta = g(\vartheta)$. In the following, a class of functions $g(\vartheta)$ of a fairly general nature (discontinuous and δ-functions $g(\vartheta)$) are sometimes permitted. Accordingly, we shall consider as solutions $x(t)$ also functions of a fairly general nature, the class of which we do not limit in the formulation of the problem. Therefore, in Definitions 1.2, 1.3 the requirement for existence of solutions is not stated even where it is necessary, and must be investigated separately, depending on the permissible class of solutions. For the same reason the proofs presented are not too rigorous assuming that, since each time the class of functions $\eta(t)$ and $x(t)$ is investigated, the appropriate calculations can be substantiated.

1.3. Let us assume (unless stated otherwise) that for each realization $\eta = g(\vartheta)$ the process of control is terminated at that moment $t = t_\epsilon[g]$ when the point on the corresponding trajectory $x(t)$ falls for the first time on the surface $\| x \|_k = \epsilon$. Therefore, in the following one can formally assume that for such realizations equalities $g(t) = 0$, $u(t) = 0$, $x(t) = 0$ are fulfilled for $t \geqslant t_\epsilon[g]$ and the quantity $p[t] = p[U, k, \epsilon, x_0, \eta_0, t_0, t]$ is a monotonic non-increasing function of time t for all $t \geqslant t_0$.

1.4. Definitions 1.2 and 1.3 include also the case of transfering $x(t)$ into the ϵ-neighborhood of the surface $l_1 x_1 + \ldots + l_n x_n = 0$, if in system (1.1) we replace the variables $\{x_i\} \to \{y_i\}$ (whereupon $y_1 = l_1 x_1 + \ldots + l_n x_n$) and insert Definitions 1.2, 1.3 $k = 1$.)

Definition 1.4. The permissible control U° (U_t° or U_k°) we shall call optimum if

$$T[U^{\circ}] = \min (T_1[U] \quad \text{for } U \in M) \tag{1.6}$$

In this article the term "optimum problem" is understood as the problem of determining the minimum of (1.6). This problem belongs to the class of problems in optimum control in the presence of a random signal at the system input [5,6]; here, however, we have some characteristic peculiarities connected with the nonlinearity of operators $u[x, \eta, t]$, $u[t, g]$ and also the fact that the problem is essentially nonstationary.*

(*Notes 1.5.* The problem is obviously generalized for the case of minimizing the functional

$$Q[U] = \int_{t_0}^{\infty} p[U, k, \varepsilon, x_0, \eta_0, t_0, t]\, L[x(t), \eta(t)]\, dt$$

1.6. Investigated analogous problems may be formulated for the case of fastest transfer of the $x(t)$ trajectory into ϵ-neighborhood of a random motion $x_i = \eta_i(t)$ $(i = 1, \ldots, n)$.

1.7. Unless stated otherwise, we shall consider only random functions $\eta(t)$ describing a random (Markov) process without consequences [7].)

2. The problem of the existence of optimum control. We shall prove the existence of optimum control U_t° for a particular case of system (1.1). Given an equation

$$x_1^{(n)} + a_1 x_1^{(n-1)} + \ldots + a_n x_1 = u_1(t) + \eta(t) \tag{2.1}$$

where the function $\eta(t)$ describes a random process of the following type: $\eta(t)$ is constant $\eta = \eta_l (l = 1, \ldots, m)$ on each semi-interval $k\tau_0 \leqslant t < (k+1)\tau_0$ ($\tau_0 > 0$ – const, $k = 0, 1, \ldots$), the probabilities of transfer $\eta_j \to \eta_l$ for $t = k\tau_0$ are constants p_{jl}, and also

$$|\eta_l| \leqslant q < 1 \qquad (l = 1, \ldots, m) \tag{2.2}$$

Unless otherwise stated, everywhere in the following it is assumed that the roots $\lambda_i (i = 1, \ldots, n)$ of the characteristic equation

$$|A - \lambda E|_1^n = 0$$

satisfy the inequality

$$\operatorname{Re} \lambda_i < -\delta \qquad (\delta > 0 - \text{const}) \tag{2.3}$$

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* It should be noted that the problems of optimum control in the presence of random disturbances are treated somewhat differently in References [17-18].

The optimum problem for Equation (2.1) at $k = n$, $\epsilon = 0$ ($x_i = x_1^{(i-1)}$, $i \neq 1$) and for the conditions (2.2) and (2.3) we shall call problem A.

Lemma 2.1. For problem A there exists a permissible control U_t regardless of the initial conditions x_0, η_0, t_0.

(*Note 2.1.* As permissible realizations $u(t)$ of control $u[t, g]$ we shall consider in the present section piece-wise smooth functions, permitting only discontinuities of the first kind at isolated values of t, as permissible solutions $x(t)$-continuous functions, satisfying Equation (2.1) for all t different from points of discontinuity $\eta(t)$ and $u(t)$.)

Proof of lemma. According to References [8,9] for the problem of optimum response for system (2.1) under conditions (2.3) $\eta(t) = 0$ and limitation

$$\|u(t)\|_1 \leqslant (1-q) \quad (u_i = 0,\ i = 2, \ldots, n) \tag{2.4}$$

there exists an optimum control $u_1 = u_1^*(t)$, regardless of the initial conditions x_0, t_0. This control (piece-wise constant function $u_1^*(t)$) brings the trajectory $x(t)$ of Equation (2.1) to point $x = 0$ ($x_1 = 0$, $x_i = x_1^{(i-1)} = 0$) at some time $t = t_0 + T^*$. Obviously, the operators

$$\begin{aligned} u[t, g] &= \{u_1^*(t) - g(t), \ldots, 0\} \quad \text{for } t_0 \leqslant t \leqslant t_0 + T^* \\ u[t, g] &= \{-g(t), 0, \ldots, 0\} \quad \text{for } t > T^* + t_0 \end{aligned} \tag{2.5}$$

constitute the permissible control U_t^* whereby

$$T[U_t^*] = T^* \tag{2.6}$$

Theorem 2.1. For problem A there exists an optimum control U_t°, regardless of the initial conditions x_0, η_0, t_0.

Proof. Given are initial conditions x_0, η_0, t_0, whereby with no loss of generality $t_0 \in [0, r_0]$. According to Lemma 2.1 there exists a permissible control U_t^*. Let us consider a sequence of permissible controls $U_t^{(k)}$ ($k = 1, 2, \ldots$), for which the following condition is satisfied

$$U_t^{(1)} = U_t^*, \qquad T[U_t^{(k)}] \geqslant T[U_t^{(k+1)}], \qquad \lim T[U_t^{(k)}] = T^\circ \quad \text{for } k \to \infty \tag{2.7}$$

and there is no permissible control U_t for which

$$T[U_t] < T^\circ \tag{2.8}$$

We shall show that there is a permissible control U_t° satisfying the condition

$$T[U_t^\circ] = T^\circ \tag{2.9}$$

Let us denote by $g_l^r(\vartheta)$ ($r = 1, 2, \ldots, l = m, \ldots, m^r, 0 \leqslant \vartheta < r\tau_0$) the realizations $\eta(t)$, and by p_l^r probabilities of g_l^r. Let us put in order the set $\{g_l^r\}$. Now, let g_l^r precede $g_{l'}^{r'}$, if $r' > r$; for fixed r, realizations g_l^r are ordered definitely on $l = 1, \ldots, m^r$. Consider triplets $\{g, u, T\}_l^r$ composed of numbers

$$T_l^r \qquad ((r-1)\tau_0 \leqslant t_0 + T_l^r \leqslant r\tau_0)$$

and functions

$$g_l^r(t), \qquad u_l^r(t) \qquad (t_0 \leqslant t < t_0 + T_l^r)$$

We shall state that the triplet $\{g, u, T\}_l^r$ belongs to $U_t^{(k)}$, if $u_1(t) = u_1[t, g_l^r] = u_l^r(t)$ and the trajectory $x(x_0, \eta_0, t_0, t, g_l^r, u[t, g_l^r])$ falls for the first time on $x = 0$ for $t = t_0 + T_l^r$. The set of all triplets $\{g, u, T\}_l^r$ belonging to $U_t^{(k)}$ we shall denote by Q_k.

Q_k does not contain simultaneously two triplets $\{g, u, T\}_l^r$ and $\{g, u, T\}_{l'}^{r'}$, where $T' > T$ and $g_l^r(t) = g_{l'}^{r'}(t)$ for $t_0 \leqslant t \leqslant t_0 + T$. Inequality

$$\sum_{(k)} p_l^r = 1 \tag{2.10}$$

is valid where the sum is taken along all realizations g_l^r contained in triplets $\{g, u, T\}_l^r \in Q_k$. Also,

$$T[U_t^{(k)}] \geqslant \sum_{(k)} p_l^r T_l^r \quad \text{for} \quad \lim\Big(\sum_{(k)} p_l^r T_l^r\Big) \leqslant T^\circ \quad \text{for } k \to \infty \tag{2.11}$$

Let us pass now to the construction of U_t°. Based on the properties of Q_k sets we can derive a diagonal subsequence (for which we will use the previous notation but renumbered) satisfying the following conditions

(1) There is a subsequence $\{g_l^r\}_k$ such that the first s terms of this subsequence are contained in all triplets Q_k, beginning with $k = s$.

(2) Equality

$$\sum_{\{g\}k} p_l^r = 1 \tag{2.12}$$

is satisfied in which the sum is taken over all elements of the sequence $\{g_l^r\}_k$ described in condition (1). (A case may arise when $\{g_l^r\}_k$ is finite, and starting with some number s, all Q_k contain a finite number of realizations g_l^r. For this case, the following discussions are simplified.) Let us denote by $(u_l^r)_k$ control functions corresponding to g_l^r from the sequence $\{g_l^r\}_k$ in control $U_t^{(k)}$, and by $(T_l^r)_k$ the numbers T contained in the appropriate triplets in Q_k. Utilising low density [10] of a unit sphere in space L_2 of function $u(t)$ ($t_0 \leqslant t < s\tau_0$, $s = 1, 2, \ldots$) for fixed s, the limitation of each sequence $(T_l^r)_k (k = 1, 2, \ldots)$, and repeating the arguments of Reference [4], one can derive a diagonal subsequence $\{Q_k\}$ (for which again we shall use the old numeration) for which the following condition is satisfied:

$$\lim (T_l^r)_k = (T_l^r)_0 \quad \text{for } k \to \infty \tag{2.13}$$

and the sequences $(u_l^r)_k$ for $k \to \infty$ slowly converge towards the limiting measurable functions $(u_l^r)^*$ on the intervals $t_0 \leqslant t \leqslant t_0 + (T_l^r)_0$. In addition, from the condition (2.11) and (2.12), it follows that

$$\sum_{\{g\}k} p_l^r (T_l^r)_0 \leqslant T^\circ \tag{2.14}$$

A limited transfer along functions $(u_l^r)_k$ slowly converging to $(u_l^r)^*$ is possible in integrals defining $x(t)$ for system (2.1) according to the Cauchy formula for inhomogeneous linear equations [11]. Therefore, the trajectories $x(t)$ of Equation (2.1) generated by the functions $(u_l^r)^*$ under the corresponding $g_l^r(t)$, arrive at the point $x = 0$ for $t = t_0 + (T_l^r)_0$.

This fact, together with conditions (2.12) and (2.14), proves that the totality of operators $u^*[t, g]$ defined on realizations g_l^r from the subsequence $\{g_l^r\}_k$ (as well as on segments $g_l^r(\vartheta)$ $(\vartheta \in [t_0, t], t \leqslant t_0 + (T_l^r)_0$ of these realizations) which compare with these realizations the functions $(u_l^r(t))^* = u_1$ for $t \leqslant t_0 + (T_l^r)_0$ and $u = -\eta(t)$ for $t > t_0 + (T_l^r)_0$ $(u_i = 0$ for $i \neq 1)$, defines the control U_l^* for which

$$T[U_l^*] = T^\circ \tag{2.15}$$

The control U_t^* is defined only in the class of measurable functions $u(t) = u^*[t, g]$.

Now, for the proof of the theorem it is sufficient to show that there exists a control U_t° also satisfying (2.15) but already in the class of the piece-wise continuous functions $u(t) = u^\circ[t, g]$.

For the proof of this assertion one should, based on the results of Reference [9], substitute in turn on the intervals $t_0 \leqslant t \leqslant r\tau_0$ $(r = 1, 2, \ldots)$ for each realization $g_l^r(t)$ the measurable functions $(u_l^r(t))^*$ into the partially constant functions $(u_l^r(t))^\circ$, $|(u_l^r(t))^\circ| \leqslant |$, which after integration, according to the Cauchy formula for solutions (2.1), on each interval, yields

$$r'\tau_0 < t < (r'+1)\tau_0, \ (r+1)\tau_0 < t < t_0 + (T_l^r)_0 \quad (r' = 0, \ldots, (r-2)) \tag{2.16}$$

the same result as the substituted functions. The existence of such partially constant functions follows from the solubility of corresponding L problems [12] on intervals (2.16) in partly constant functions

$(u_l{}^r(t))^\circ$, if these problems are soluble in measurable limited functions $(u_l{}^r(t)^*$ on the same intervals. The theorem is proved.

(*Note* 2.2. Analogously, based on the lemma from Reference [13] (p. 575) one can prove the existence of optimum control $U_t{}^\circ$ for which the control functions assume only two values $u = +1$ and $u = -1$.)

3. Necessary conditions of optimal control. In this section the necessary conditions are derived optimising U_t for a problem similar to A. This problem is "smoothed" by introduction of an additional random quantity ξ with a small dispersion σ^2. The conditions for optimization of A can be obtained for the optimization conditions for this problem by letting $\epsilon \to 0$ and $\sigma^2 \to 0$; substantiation of this limiting transfer is, however, beyond the scope of the present discussion. Note that the analogous introduction of ξ may be applied for derivation of optimization conditions in a stable system in the absence of the random disturbance $\eta(t)$, i.e. for the ordinary problem of high speed response. Consider the equation

$$x_1{}^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x_1 = u(t) + \eta(t) + \xi\delta(t - t_0) \qquad (3.1)$$
$$(x_i = x_1{}^{(i-1)};\ i = 1, \ldots, n)$$

where ξ is a random independent quantity with a normal distribution $M\{\xi\} = 0$, $M\{\xi\} = \sigma^2$ (here and in the following the symbol $M\{a\}$ denotes the mathematical expectancy of a). For Equation (3.1) we shall consider the following problem in optimization:

Problem B. It is required to determine the optimum ϵ-permissible t-control $U_t{}^\circ\{u^\circ[t, g]\}$ on the coordinate x_1 under the condition that in the control process the value of the random quantity ξ remains unknown, and consequently, as in Definitions 1.2 and 1.4, the control $u^\circ[t, g]$ is based only on the information about realization $g(\vartheta)$ of a random function $\eta(t)$ for $\vartheta \leqslant t$ ($\eta(t)$ function is described in Section 2).

In the present section we shall assume that for all realizations g the operators $u^\circ[t, g]$ are defined for all $t \geqslant t_0$. Renumerate, consequently, the realizations $g(\vartheta)$ in each of the intervals

$$r\tau_0 \leqslant t < (r+1)\tau_0 \qquad (r = 0, 1, \ldots) \qquad (3.2)$$

as follows. In each interval $0 < \vartheta < r_0$ let all realizations $\eta = g(\vartheta)$ be renumerated by indices $l_0 = 1, \ldots, m$ and denoted by $g_{l_0}(\vartheta)$, and corresponding probabilities denoted by p_{l0}. Each $g_{l0}(\vartheta)$ generates in the interval $r_0 < \vartheta < 2r_0$ m-realizations $\eta = g(\vartheta)$ which we will denote by the symbols $g_{l0\,l_1}(\vartheta)$ ($l_0 = 1, \ldots, m$; $l_1 = 1, \ldots, m$). By induction let us renumerate $g(\vartheta)$ in all intervals of (3.2). Let the initial value be $t_0 \in [0, r_0)$ and $\eta_0 = g_{l_0'}(t_0)$, then the probability

$$p[U_t^\circ, 1, \epsilon, x_0, \eta_0, t_0, t] = p(t)$$

of inequality $|x_1(x_0, \eta_0, t_0, t, \eta, u)| > \epsilon$ for $t \in (r\tau_0, (r+1)\tau_0)$ in the given case is calculated from

$$p[t] = \sum_{(l_1 \ldots l_r)} p_{l_0' l_1 \ldots l_r}\Big(1 - \int_{-\varepsilon}^{\varepsilon} (2\pi)^{-1/2}\gamma(t)^{-1}[\exp -(\zeta - \zeta[l_0', l_1 \ldots l_r, t])^2 (2\gamma^2(t))^{-1}] d\zeta\Big) \tag{3.3}$$

where the summation is overall $l_1, \ldots, l_r$ ($l_j = 1, \ldots, m$) for fixed l_0', and the quantities γ and $\zeta[l_0', l_1, \ldots, l_r, t]$ are determined by the equalities

$$\gamma^2(t) = \sigma^2 h_{1n}^2(t - t_0) \tag{3.4}$$

$$\zeta[l_0', l_1 \ldots l_r, t] = \sum_{j=1}^{n} h_{1j}(t - t_0) x_{j0} + \tag{3.5}$$

$$+ \sum_{k=1}^{r} \int_{k\tau_0}^{(k+1)\tau_0} h_{1n}(t - \vartheta)[u^\circ[\vartheta, g_{l_0' l_1 \ldots l_k}] + g_{l_0' l_1 \ldots l_k}(\vartheta)] d\vartheta +$$

$$+ \int_{t_0}^{\tau_0} h_{1n}(t - \vartheta)[u^\circ[\vartheta, g_{l_0'}] + g_{l_0'}(\vartheta)] d\vartheta +$$

$$+ \int_{r\tau_0}^{t} h_{1n}(t - \vartheta)[u^\circ[\vartheta, g_{l_0' l_1 \ldots l_r}] + g_{l_0' l_1 \ldots l_r}(\vartheta)] d\vartheta$$

In the equalities (3.4) and (3.5) functions $h_{ij}(t)$ are the elements of the fundamental matrix of solutions for the homogeneous system (3.1) ($h_{ij}(t_0) = \delta_{ij}$). Thus the optimal problem B is reduced to minimization of the functional

$$T[u_t^\circ] = \int_{t_0}^{\infty} p[t] dt = \min \tag{3.6}$$

where the probability $p[t]$ is determined by the equalities (3.3) to (3.5). This problem can be solved by the usual methods of variational calculus. In particular, let $\delta u[t, g]$ be the variation of control $u[t, g]$, permissible under condition (1.2), equal to zero for all realizations $g(\vartheta)$ except the marked realization $g^*(\vartheta)$ for which it is also equal to zero everywhere except at some small interval $(t^* - a, t^* + a)$, located fully inside the interval $r\tau_0 < t < (r+1)\tau_0$. At the same time we shall assume that in the described numeration above the segment $g^*(\vartheta)$ in interval $\vartheta \in (r\tau_0, (r+1)\tau_0)$ is denoted by $g_{l_0' l_1' \ldots l_r'}$. Then the sign of the variation δT will be defined by the sign of the expression

$$f(t^*)\,\mathrm{sign}\,\delta u_1(t^*) = \tag{3.7}$$

$$= \mathrm{sign}\,\delta u_1(t^*)\Big[\int\limits_{t^*}^{(r+1)\tau_0} p_{l_0',l_1',\ldots,l_r'}F(l_0',l_1',\ldots,l_r',t)\,h_{1n}(t-t^*)\,dt +$$

$$+\sum_{k=r+1}^{\infty}\int\limits_{k\tau_0}^{(k+1)\tau_0}\sum_{(l_{r+1},\ldots,l_k)} p_{l_0',l_1',\ldots,l_r',l_{r+1},\ldots,l_k}F(l_0',\ldots,l_r',l_{r+1},\ldots,l_k,t)\,h_{1n}(t-t^*)\,dt\Big]$$

where

$$F[l_0',\ldots,l_r',\;l_{r+1},\ldots,l_k,t]=\frac{-1}{\sqrt{2\pi\gamma^3(t)}}\times \tag{3.8}$$

$$\times\int\limits_{-\varepsilon}^{\varepsilon}\exp\frac{-(\zeta-\zeta[l_0',\ldots,l_r',l_{r+1},\ldots,l_k,t])^2}{2\gamma^2(t)}\,[\zeta-\zeta[l_0',\ldots,l_r',l_{r+1},\ldots,l_k,t]]\,d\zeta$$

for $k\,\tau_0 < t < (k+1)\,\tau_0$.

Let us consider function $f(t^*)$, defined by the equality (3.7). For interpretation of this function let us consider the system (1.1) which is equivalent to Equation (2.1), where consequently $x = \{x_1, \ldots, x_n\} = \{x_1, \ldots, x_1^{(n-1)}\}$:

$$A=\begin{Vmatrix} 0 & 1 & 0 & \ldots & 0\\ 0 & 0 & 1 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & \ldots & 1\\ -a_n & -a_{n-1} & 0 & \ldots & -a_1\end{Vmatrix},\qquad B=\begin{Vmatrix} 0 & 0 & \ldots & 0\\ 0 & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & 0\\ 0 & 0 & \ldots & 0\\ 1 & 0 & \ldots & 0\end{Vmatrix}$$

According to known properties of linear systems, [11] functions $h_{1i}(t - t^*)$ $(i = 1, \ldots, n)$ of argument t^* constitute a particular solution of the system

$$d\psi/dt^* = -A^*\psi \tag{3.9}$$

comparable to system (1.1) (A^* is the transposed matrix A). Consequently, function $h_{1n}(t - t^*)$ may be considered a scalar product of the solution vector $\{h_{1i}(t - t^*)\}$ and the vector b is the first column of matrix B. Applying the rules for differentiation of integrals in (3.7) with respect to t^* and taking into consideration the conditions $h_{1i}(0) = 0$, $h_n(0) = 1$ for $i \neq 1$, we verify that the function $f(t^*)$ may be considered as a scalar product $f(t^*) = (b \times \psi(t^*))$, where $\psi(t^*)$ is the solution of the system

$$\frac{d\psi}{dt} = -A^*\psi + d(t),\qquad d(t) = \{-pF(t), 0, \ldots, 0\} \tag{3.10}$$

When $T[u_t^0]$ is minimum, the variation should not be negative, consequently the following fact may be established.

Theorem 3.1. At each interval $k\tau_0 < t < (k+1)\tau_0$ the optimum control $u[t, g]$ for each $g(\vartheta)$ in problem B satisfies the condition: the operator $u^\circ[t, g]$ is such that the quantity

$$-u_1^\circ[t,g]f[t] = -u_1^\circ[t,g](b\cdot\psi(t)) = \max \tag{3.11}$$

where $\psi(t)$ is the solution of system (3.10) (for $t = t^*$).

(*Note 3.1.* Condition (3.11) corresponds in the case considered to the principle of maximum [4].)

In passing from one realization to another and in passing through the critical values $t = k\tau_0$, the solutions $\psi(t)$ (3.9), defining according to (3.10) and (3.11) the optimum control $u[t, g]$, deviate. However, one can see from formulas (3.3) to (3.8) that the deviations of $\psi(t)$ in passing through the point $t = k\tau_0$ are subject to the rule of discontinuity: let $\psi_{l_0', \ldots, l_k'}(t)$ (the defining solution on $g_{l_0' \ldots l_k'}$ for t in the interval $(k\tau_0, (k+1)\tau_0)$ and $\psi_{l_0', \ldots, l_k', l_{k+1}}(t)$) be the defining solutions on $g_{l_0', \ldots, l_k' l_{k+1}}$ for t in the interval $((k+1)\tau_0\ (k+2)\tau_0)$; then the following equality holds*

$$\lim_{t\to(k+1)\tau_0-0}(\psi_{l_0',\ldots,l_k'}(t)) = \sum_{l_{k+1}=1}^{m} P_{l_{k+1}} \lim_{t\to(k+1)\tau_0+0}(\psi_{l_0',\ldots,l_k',l_{k+1}}(t)) \tag{3.12}$$

(*Note 3.2.* The method described for smoothing the optimal problem lends itself to the application of one of the direct methods for solution in variational problems, for example, the method of fastest descent for calculation of optimum control. The situation is analogous to that occurring in solutions by direct methods of the known optimal problems (see for example [5,6,14]). However, it should be noted, that in minimizing the functional (3.6) the resulting computational difficulties are quite considerable.

Verification of the existence of a permissible and optimal control and the construction of optimum control in concrete cases is difficult. The

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* It may be considered that this discontinuity is defined by terms of the δ-function type in the right hand side of (3.10), corresponding to the transfer matrix $\eta_i \to \eta_j$. For the case of continuous process $\eta(t)$, this fact causes the corresponding appearance of the continuous members in the right hand side of (3.10). It is worth noting that the appearance of $d(t)$ in (3.10) is due here to the introduction of ξ and for $\sigma^2 \to 0$ function $d(t) \to 0$ outside the neighborhood of point $x = 0$.

possibility of applying the method of Liapunov functions is investigated in the following section [1,2,15]. The application of this method to control problems was developed by Chetaev. In particular, the problem of parameter selection in a stable system for optimum high speed response was solved for linear systems by Chetaev in References [2,15] based on Liapunov quadratic form functions. In addition, Reference [15] derived concrete evaluations of decay times for transient processes up to the given quantity $\epsilon > 0$ based on the characteristic numbers of the Liapunov function v of quadratic form and its derivative dv/dt, on the strength of the equations for a disturbed motion.)

4. Application of the method of Liapunov functions to the optimal problem. In this section a generalization of Liapunov functions is described which permits the application of these functions as apparatus for the investigation of high-speed response problems in control systems, including the presence of random disturbances. The application of the second method of Liapunov to problems of system response in the absence of random disturbances is described in Reference [9]. Note that the surfaces of the optimum level of Liapunov functions considered in Reference [9] are apparently isochronisms in the sense of Reference [16]. The author considers it his duty to point out that the discussions given in this section overlap in some essential respects with the researches of Repin, who has worked out a method for solving optimal problems on the basis of the methods of dynamic programming and derived a general partial differential equation for the minimizing functional.

Let us first introduce a number of definitions corresponding in our case to the classical definitions of the second method of Liapunov [1,2].

We will consider functions $v(x, \eta, t)$ of coordinates $x_i (i = 1, \ldots, n)$ for a random value η and time t, not assuming them to be continuous for all arguments.

Definitions 4.1. We will call the function $v(x, \eta, t)$ positive definite in region $G \times H$ for $t \geqslant t_0$, if the following condition is satisfied

$$v(x, \eta, t) > 0 \quad \text{for } x \in G, \ x \neq 0, \ \eta \in H, \ t \geqslant t_0 \tag{4.1}$$

4.2. The function $v(x, \eta, t)$ admits an infinitely small higher limit (in $G \times H$ for $t \geqslant t_0$), if there is a constant L satisfying the condition

$$v(x, \eta, t) \leqslant L \|x\| \quad \text{for } x \in G, \ x \neq 0, \eta \in H, \ t \geqslant t_0 \tag{4.2}$$

4.3. In substituting in $v(x, \eta, t)$ for x_i and η, the coordinates $x_i(t)$, the solution of system (1.1) (corresponding to some control U_t (or U_x) and the values of the random function $\eta(t)$, one obtains the random functions of time $v(t)$. Let us assume that for the mathematical expectancy $M\{v(t)\}$ of this function one can compute the right derivative $M\{v\}$.

We will say that the function $v(x, \eta, t)$ has a negative definite derivative $dM\{v\}/dt$ (in the region $G \times H$ for $t > t_0$), if the inequality

$$dM\{v\}/dt \leqslant -\delta \qquad \text{for } x \in G,\ \eta \in H,\ t \geqslant t_0 \qquad (\delta = \text{const} > 0) \quad (4.3)$$

is satisfied and if (4.3) can be integrated, i.e.,

$$M\{v(t)\} - M\{v(t_0)\} \leqslant -\delta(t - t_0) \qquad (t \geqslant t_0) \quad (4.4)$$

The last restriction is necessary, since some more general functions are allowed as Liapunov functions $v(x, \eta, t)$ and solutions $x(t)$ than those from the classical cases of the Liapunov theory.

4.4. The function $v(x, \eta, t)$ satisfying the conditions of 4.1 to 4.3 we shall call the generalized Liapunov function (in the corresponding region).

(*Notes 4.1*. In this section we will consider only optimal problems for $\epsilon = 0$, $k = n$, $G = \{-\infty < xi < \infty\}$. According to Note 1.3, each realization of the trajectory $x(t)$ is defined by the given control U_t(or U_x) only for $x \neq 0$ (for $\epsilon = 0$), and after reaching the point $x = 0$ for $t = t^*$ we have $x(t) \equiv 0$. Accordingly, we will assume that for $x = 0$ Liapunov functions are not defined and that each realization of the function $v(t)$ after reaching the corresponding realization of solution $x(t)$ at point $x = 0$ is continued in such a manner that $dv/dt = -1$.

4.2. In what follows we will consider only those cases when the random solutions $x(t)$ of system (1.1) have for all $t > t_0$ a finite dispersion uniformly bounded at $t > t_0$. In consequence of (2.3) this condition is satisfied, for example, if functions $\eta(t)$ and $u[t, \eta]$ possess a finite dispersion uniformly bounded for $t \geqslant t_0$).

Lemma 4.1. If for a given control U_t(or U_x) for the system (1.1) we can give a generalized Liapunov function $v(x, \eta, t)$, then the control U_t(or U_x) is permissible.*

Proof. Due to imposed conditions, $x(t)$ has bounded dispersion. Consequently, $v(t)$ due to the infinitely small higher limit (4.2) will also have a bounded dispersion $D(t) < D$ = const. for all $t > t_0$. Because

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* See Note 1.2. The control U_t is here assumed such that $\{x(t), \eta(t)\}$ is a random process without after-effects.

$v(x, \eta, t)$ is positive definite (4.1), the probability $p[t]$ of inequality $x(t) \neq 0$ coincides with the probability of inequality $v(t) > 0$. From the condition (4.4) according to Chebyshev's inequality [7] (p. 187) we conclude now that the probability $p[t]$ is of the order $1/t^2$ for $t \to \infty$. This proves the convergence of integral (1.3).

Definition 4.5. The generalized Liapunov function $v^\circ(x, \eta, t)$ will be called optimal if the condition is satisfied that

$$\min_U dM\{v^\circ\}/dt = -1 \tag{4.5}$$

at each point x, η (or correspondingly at each instant t of the control process).

Theorem 4.1. Let system (1.1) possess an optimal control U_x° (for U_t° for all initial conditions $x, \eta, t \geqslant t_0$). Then the positive definite function $v(x, \eta, t) = T[U, n, 0, x, \eta, t]$ satisfies (4.5), whereupon the minimum is reached on the optimum control U_x° (or U_t°).

Proof. Compute $dM\{v\}/dt$ for the function $v = T[U^\circ]$ at optimum control U° (at the point $x = x_0$, $\eta = \eta_0$, $t = t_0$). Quantity $p[U, k, \epsilon, x(t_0 + \tau), \eta(t_0 + \tau), t_0 + \tau, t]$ for fixed U and for constant $t \geqslant t_0 + \tau$ is a random function τ, the statistical properties of which are determined by $x(t)$ and $\eta(t)$. By definition of $M\{v(t)\}$ we have

$$M\{v(t_0 + \Delta t) - \{v(t_0)\}\} =$$

$$= M\left\{\int_{t_0+\Delta t}^{\infty} p[U, n, 0, x(t_0 + \Delta t), \eta(t_0 + \Delta t), t_0 + \Delta t, t]\,dt\right\} -$$

$$-\int_{t_0}^{\infty} p[U, n, 0, x_0, \eta_0, t_0, t]\,dt$$

and in view of the known properties of the random processes without after effects [7] we have

$$M\left\{\int_{t_0+\Delta t}^{\infty} p[U, n, 0, x(t_0 + \Delta t), \eta(t_0 + \Delta t), t_0 + \Delta t, t]\,dt\right\} =$$

$$= \int_{t_0+\Delta t}^{\infty} p[U, n, 0, x_0, \eta_0, t_0, t]\,dt$$

i.e.

$$\left(\frac{dM\{v\}}{dt}\right)_{U_0} = \left(\frac{d}{dt'}\left[\int_{t'}^{\infty} p[U, n, 0, x_0, \eta_0, t_0, t]\,dt\right]\right)_{t'=t_0} = -1 \tag{4.6}$$

since when $x_0 \neq 0$ on some sufficiently small interval $(t_0, t_0 + \Delta t)$

$$\lim p[U, n, 0, x_0, \eta_0, t_0, t] = 1 \text{ при } \Delta t \to 0$$

Let us assume now that for some permissible control U at separate times of control $dM\{v\}/dt_0 < -1$. Then on some interval Δt the inequality $M\{v(t_0 + \Delta t)\} - M\{v(t_0)\} < -\Delta t$ would be satisfied which in consequence of (4.6) contradicts the assumption regarding the optimum of control U°.

The obtained contradiction along with the equality (4.6) proves the theorem.

Theorem 4.2. Let there be given a generalized optimal $v(x, \eta, t)$ Liapunov function for system (1.1). If for some control $U^\circ(U_t^\circ$ or $U_x^\circ)$ this function satisfies the condition $dv^\circ/dt = -1 = \min$, then this control U° is optimal.

Proof. According to Lemma 4.1 the control U° is permissible. Let us assume the contrary, namely, that this control U° is not optimal, and consequently there is a permissible control U^* for which

$$T[U^*] < T[U^\circ] \tag{4.7}$$

at least for one point x_0, η_0, t_0, whereby

$$\left(\frac{dM\{v\}}{dt}\right)_{U^*} \geqslant -1 \tag{4.8}$$

Let us consider the process of control under the given initial conditions x_0, η_0, t_0, and let us denote by v_t° and $v_t^{\circ *}$ the corresponding mathematical expectancies (computed at $t = t_0$) of the random functions $v^\circ(t)$ and $v^{\circ *}(t)$, corresponding to controls U° and U^*, for those values $x(t)$ and $x^*(t)$ for which $v^\circ(t) > 0$ and $v^{\circ *}(t) > 0$. From conditions $(dM\{v^\circ\}/dt)_{U^\circ} = -1$ and $dv^\circ/dt = -1$ at $x = 0$, and also from (4.8) follow the conditions

$$\left(\frac{dv_t^\circ}{dt}\right)_{dt=+0} = -p[U^\circ, n, 0, x_0, \eta_0, t_0, t] \tag{4.9}$$

$$\left(\frac{dv_t^{\circ *}}{dt}\right)_{dt=+0} \geqslant -p[U^*, n, 0, x_0, \eta_0, t_0, t]$$

(We assume that these derivatives are meaningful and that relations (4.9) are integrable).

From conditions (4.9) it follows that

$$T[U^\circ] = v^\circ(t_0), \qquad T[U^*] \geqslant v^\circ(t_0) \tag{4.10}$$

which contradict (4.7).

Theorems 4.1 and 4.2 indicate the application of the second method of Liapunov to optimal problems. It should be noted, however, than an effective construction of the optimum Liapunov function $v(x, \eta, t)$ is difficult. If one assumes that in the neighborhood of some point x, η, t the function v is differentiable with respect to x_i and t, then condition (4.5) leads to a partial differential equation which must be satisfied by the generalized Liapunov function. For example, if the random function $\eta(t)$ assumes m values $\eta_1, \ldots, \eta_m$, satisfying the limitation (2.2) whereby the probability $p_{lj}(\Delta t)$ of the transformation $\eta_l \to \eta_j (l \neq j)$ on the interval $(t, t + \Delta t)$ is determined by the conditions

$$p_{lj}(\Delta t) = p_{lj}\Delta t + o(\Delta t) \qquad (p_{lj} = \text{const}) \tag{4.11}$$

then condition (4.5) leads to the equalities

$$\min \Big[\sum_{i=1}^{n} \frac{\partial v^\circ(x, \eta_l, t)}{\partial x_i} \Big[\sum_{j=1}^{n} (a_{ij}x_j + b_{ij}u_j) + c_i\eta_l \Big] + \frac{\partial v^\circ}{\partial t} + \tag{4.12}$$
$$+ \sum_{k \neq l} p_{lk} [v^\circ(x, \eta_k, t) - v^\circ(x, \eta_l, t)] \Big] = -1 \qquad \text{при } \|u\| \leqslant 1 \quad (l = 1, \ldots, m)$$

which can be considered as a system of m equations for m functions $v_l = v^\circ(x, \eta_l, t)$. The difficulty of defining v_l is connected, in particular, with the circumstance that it is required to find positive definite solutions v_l of the system (4.12) for $x \neq 0$.

In Section 5 we will consider a case when the solution of the problem is facilitated by the fact that the optimal function $v^\circ(x, \eta, t)$ does not depend explicitly on η_l. In Section 6 an approximate graphical method for constructing functions $v^\circ(x, \eta, t)$ for a second order system is described.

5. Optimal control for the case of white noise at system input. Let the random function $\eta(t)$ describe white noise which we shall assume realized as a limiting case of a shot effect (random impulses with dispersion a^2 and mean density ν distributed an axis t) for $\nu \to \infty$ and $a^2\nu = \gamma = \text{const}$ [6]. Assume that $M\{\eta(t)\} = 0$. Since the considered random process $\eta(t)$ is white noise, the information about past realization of the signal does not play a role in the choice of control for the future, i.e. at each instant t and point $x(t)$ the U_x should be chosen by the same rule as the optimal control in the absence of a random signal $\eta(t)$.

Let us consider the application of Liapunov functions for this case. Let matrix B in system (1.1) not be singular. It was shown in Reference [9] that in this case for $\eta = 0$ there exists an optimal function of Liapunov $v^\circ(x)$, defined for all x and possessing for $x \neq 0$ continuous partial derivatives. (In Reference [9] condition (4.2) is not proved for $v^\circ(x)$ but this condition can be verified here). The optimum control U°_x corresponds to the function $v^\circ(x)$. If one computes $dM\{v^\circ\}/dt$ for the function $v^\circ(x)$, then, on the strength of system (1.1), in the presence of a random signal $\eta(t)$ and permissible control U, we will have

$$\left(\frac{dM\{v^\circ\}}{dt}\right)_U = \sum_{i=1,\ j=1}^{n} \frac{\partial v^\circ}{\partial x_i}[a_{ij}x_j + b_{ij}u_j] + \\ + \lim_{\Delta t \to +0}\left[\frac{1}{\Delta t}\left(M\{v(x_\eta(t+\Delta t))\} - v(x(t+\Delta t))\right)\right] \tag{5.1}$$

where $x_\eta(t)$ is the random function $x(t)$ (1.1), generated by random $\eta(t)$ and $u(t)$.

The first sum is of the same form as in the case of $\eta(t) = 0$. Utilizing the properties of a shot effect, passed through a linear filter (1.1) [6], the equality $M\{\eta(t)\} = 0$ as well as property (4.2) and the ability of continuous differentiation of $v^\circ(x)$, we can verify that the second sum on the right-hand side of (5.1) is equal to zero.

6. In this section an approximate graphical method is described for constructing an optimal Liapunov function $v(x, y)$ for a second order system (x, y-scalars). We shall follow the geometrical interpretation of the Liapunov function indicated by Chetaev.

Let us consider the second order system

$$\frac{dx}{dt} = \mu y + (1-\mu)u_2, \qquad \frac{dy}{dt} = -\mu(a_2 x + a_1 y) + u_1 + \mu\eta(t) \tag{6.1}$$

corresponding to the optimal problem for equation

$$\ddot{x} + a_1\dot{x} + a_2 x = u_1 + \eta(t) \tag{6.2}$$

under conditions

$$|u_1| \leqslant 1 \quad \text{for} \quad \mu = 1 \tag{6.3}$$

Consider only the case when $\eta(t)$ can assume two values η_1 and η_2, and the probability $p_{lj}(\eta_l \to \eta_j)$ on the interval $(t,\ t+\Delta t)$ is of the form: $p_{lj} = p\Delta t + 0(\Delta t)$ (p = const).

Assume that the optimal generalized function $v^\circ(x, \eta, \mu)$ depends continuously on the parameter μ. In Reference [9], the continuous dependence of v° on μ is proved for $\eta = 0$. Here we take this fact as a hypothesis. This is sensible, since in the course of construction the generalized Liapunov function v is obtained in any case such that it ensures the passing of the trajectory through the point $x = y = 0$ with large probability for large t (see Lemma 4.1). The sufficiency of the obtained response can be verified by the specific conditions of the problem.

Let us describe the construction of function v°. Divide the interval $0 < \mu < 1$ into n parts at points $\mu_0 = 0, \mu_1, \ldots, \mu_n = 1$. For $\mu_0 = 0$ values $v^\circ(x, y, \eta_1, \mu_0)$ and $v^\circ(x, y, \eta_2, \mu_0)$ coincide and the level lines for function $v^\circ(x, y, \eta_l, \mu_0)$ are constructed by elementary means (see [9]).

Assume that on the surface xy in region D of possible deviations x, y, there are constructed level lines for functions $v^\circ(x, y, \eta_1, \mu_i)$ and $v^\circ(x, y, \eta_2, \mu_i)$. Let these level lines be constructed for values v° = const $= k r_0$, where r_0 is a sufficiently small positive constant $k = 1, 2, \ldots$ Assume also that for function $v^\circ(x, y, \eta_1, \mu_{i+1})$ there are constructed level lines* $v^\circ = j r_0 (j = 1, \ldots, m)$. We will describe the construction of level lines for

$$v^\circ(x, y, \eta_1, \mu_{i+1}) = (m+1)\tau_0$$

If it is assumed that the function $v^\circ(x, y, \eta_1, \mu_{i+1})$ in the neighborhood of point (x, y) is differentiable, then according to (4.12), at this point the following conditions must be satisfied

$$\left(\frac{dM\{v^\circ\}}{dt}\right)_{U^\bullet} = \min_{\|u\|\leqslant 1}\left[\frac{\partial v^\circ}{\partial x}\left[\mu_{i+1}y + (1-\mu_{i+1})u_2\right] + \frac{\partial v^\circ}{\partial y}\left[-\mu_{i+1}(a_2 x + a_1 y - \eta_1) + u_1\right] + p\left[v^\circ(x, y, \eta_2, \mu_{i+1}) - v^\circ(x, y, \eta_1, \mu_{i+1})\right]\right] = -1 \qquad (6.4)$$

We shall assume, in accordance with out hypothesis, that a small change $\Delta\mu = \mu_{i+1} - \mu_i$ causes a small change in v°. Having chosen a point x_0, y_0 on the line $v^\circ(x, y, \eta_1, \mu_{i+1})$ it is possible to obtain a point x_1, y_1, located on the line $v^\circ(x, y, \eta_1, \mu_{i+1}) = (m+1) r_0$. The coordinates x_1, y_1 are computed from

$$x_1 = x_0 + \Delta x_1, \qquad y_1 = y_0 + \Delta y_1 \qquad (6.5)$$

where

$$\Delta x_1 = -\tau_0(\mu_{i+1} y_0 + (1-\mu_{i+1})u_2^\circ)/\beta$$

$$\Delta y_1 = -\tau_0(-\mu_{i+1}(a_2 x_0 + a_1 y_0 - \eta_1) + u_1^\circ)/\beta \qquad (6.6)$$

$$\beta = -1 - p\left[v^\circ(x_0, y_0, \eta_2, \mu_i) - v^\circ(x_0, y_0, \eta_1, \mu_i)\right]$$

* Level lines for $v^\circ = r_0$ for all v° may be constructed assuming $x = y = 0$ on the right-hand side of system (6.1) and neglecting the transformation of η. Then the problem of line construction $v = r_0$ is solved elementarily.

and numbers $u_1°$ and $u_2°$ are chosen by the condition that the scalar product of vectors $\{u_1°, u_2°(1-\mu_i)\}$ and $\{\partial v°(x_0, y_0, \eta_1, \mu_i)/\partial y, \partial v°(x_0, y_0, \eta_1, \mu_i)/\partial x\}$ at point x_0, y_0 is a minimum with respect to u_1 and u_2, $u_1^2 + u_2^2 < 1$. Having chosen a sufficiently large number of points (x_0, y_0) on the curve $v° = mr_0$, we shall obtain by this means sufficiently many points $(x_1, y_1) \in \{v° = (m+1)r_0\}$. Connecting these points by a smooth curve, we will obtain* the line $v° = (m+1)r_0$. Upon construction of the necessary number of level lines of $v°(x, y, \eta_1, \mu_{i+1})$ one can analogously construct the level lines for $v°(x, y, \eta_2, \mu_{i+1}) = \text{const}$. Then we proceed to the construction of level lines for $v°(x, y, \eta_1, \mu)$ and $v°(x, y, \eta_2, \mu)$ for $\mu = m+2$ etc. up to the value $\mu = \mu_n = 1$. This concludes the construction of level lines for $v°(x, y, \nu_1, 1)$ and $v°(x, y, \eta_2, 1)$ on the surface xy. Having obtained the plot of these level lines, one can construct the switching line for the control function $u_1(t)$. Indeed, we will assume that the level lines for $v°(x, y, \eta_1, 1)$ are constructed on the first sheet of x, y surface, and the level lines for $v°(v, y, \eta_2, 1)$ on a second one. The curves on each sheet, connecting the points where the tangent to the level lines is parallel to axis Oy, will be the curves of switching: function $u_1(x, y)$ will change sign only in passing through these curves (on a given sheet i.e., for given $\eta = \eta_l (l = 1, 2)$). Function $u_1(t)$ will also change sign during the change of values η_l, when the representing point (x, y) passes from one sheet of surface xy to another one, whereby the surfaces xy on the various sheets correspond to this point in the regions of different signs of function u.

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* Here we are not considering the possibility of the line $v° = \text{const}$ crossing itself and other complications which may arise in the course of construction.

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